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Euler Characteristic in Percolation Theory

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The notion of the Euler characteristic is introduced in percolation theory, which, in fact, was implicitly used from the very beginning of studying percolation problems. An exact formula is given for the case of the ball problem, along with some of its generalizations.

KEY WORDS:

One of the key problems of percolation theory is to determine the number of clusters or, speaking in terms of algebraic topology, the zero Betti number, as a function of the percolation parameter. However, this problem seems to be very difficult, as no exact formulas have been obtained up to now. So it would be of interest to calculate other possible topological invariants. One such invariant is the Euler characteristic. Its calculation for two-dimensional lattice problems was performed in fact by Sykes and Essam.[1] For certain lattices they constructed a special "matching" polynomial, which is nothing else but the Euler characteristic of the colored part of the plane. For some special lattices these polynomials gave them the possibility to calculate the exact values of percolation thresholds. Recently Bardeen *et al.*[2] and independently Hamilton *et al.*[3] obtained an expression for the genus of the boundary surface for the continuous percolation problem. It is easy to see that the genus they used is again the Euler characteristic of the colored phase taken with a negative sign. Here I present a calculation of the Euler characteristic for the ball problem of percolation theory.

To define the ball problem, consider an independent uniform random distribution of points in the d-dimensional Euclidean space with the mean

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density ρ. Surround every point by a d-dimensional ball B^d of radius r centered at this point. We are interested in the topological properties of the union of all balls.

The main topological characteristics of a compact figure are the so-called Betti numbers β_i, $0 \leqslant i < d$ (the ranks of the corresponding homology groups), which have the intuitive meaning of the number of cavities with i-dimensional boundary in the figure. In particular, β_0 is the number of separate connected components (clusters) of the figure, and β_1 is the number of mutually noncontractible loops which can be drawn on the figure.

In our case the figure is unbounded, so we choose specific Betti numbers per point. More precisely, let us consider the large but bounded volume V. This volume cuts out a compact part of our figure. The specific Betti numbers are the limits of the ratios of the Betti numbers of this part to the number of the points getting in V when V tends to the whole space. It is easy to see that such "specific topology" depends not on r and ρ separately, but only on their dimensionless combination:

$$\alpha = V_d(r)\rho$$

where $V_d(r)$ is the volume of the d-dimensional ball of radius r. According to this definition α is the mean number of points in one ball of radius r.

Let us define the specific Euler characteristic as the alternating sum of specific Betti numbers

$$\chi_d(\alpha) = \beta_0(\alpha) - \beta_1(\alpha) + \cdots + (-1)^{d-1} \beta_{d-1}(\alpha)$$

The main result of this paper is the following.

Theorem. $\chi_d(\alpha)$ are given by the following recurrence formulas:

$$\chi_1(\alpha) = e^{-\alpha}$$

$$\chi_d(\alpha) = \frac{d}{d\alpha}\,[\alpha\chi_{d-1}(\alpha)]$$

In particular,

$$\chi_2(\alpha) = (1-\alpha)\,e^{-\alpha}$$

$$\chi_3(\alpha) = (1-3\alpha+\alpha^2)\,e^{-\alpha}$$

Proof. First notice that balls forming the figure give a covering; by means of its nerve $\mathcal{N}$ we can calculate the homology of the figure.[2] Let us

[2] The nerve of the covering is a polyhedron whose n simplexes corresponds to $(n+1)$-multiple intersections of elements of the covering. For sufficiently "good" covering the nerve and the figure are homotopically equivalent. See ref. 4 for details.

denote by $C_{n,d}(\alpha)$ the specific number of the n-simplexes of the nerve [i.e., the specific number of the $(n+1)$-multiple intersections of the balls]. In particular, $C_{0,d}(\alpha)$ is the specific number of balls themselves, so $C_{0,d}(\alpha) \equiv 1$. The specific variant of the classical Euler–Poincaré theorem gives the equality

$$\chi_d(\alpha) = \sum_{n=0}^{\infty} (-1)^n C_{n,d}(\alpha)$$

So, to determine $\chi_d(\alpha)$, it is sufficient to calculate $C_{n,d}(\alpha)$.

Lemma:

$$C_{n,d}(\alpha) = \frac{(n+1)^{d-1}}{n!} \alpha^n$$

Proof. First consider the case $n=1$. It is easy to see that two vertices form an edge in the nerve $\mathcal{N}$ if and only if one of them lies in the ball of radius $2r$ with center at the other vertex. So when $V \to \infty$ we have

$$C_{1,d}(\alpha) = (1/2!)[2^d V_d(r)\rho]\, V\rho/V\rho = 2^{d-1}\alpha$$

Consider now the case of arbitrary n. Denote by $T = (B^d)^{n+1} \subseteq \mathbb{R}^{d(n+1)}$ the direct product of $n+1$ d-dimensional balls centered at 0. Let L be the d-dimensional diagonal $x_1^{(d)} = x_2^{(d)} = \cdots = x_{n+1}^{(d)}$ and let F be the strip obtained by moving T along L (see Fig. 1 for $d=1$, $n=1$). It is easy to see

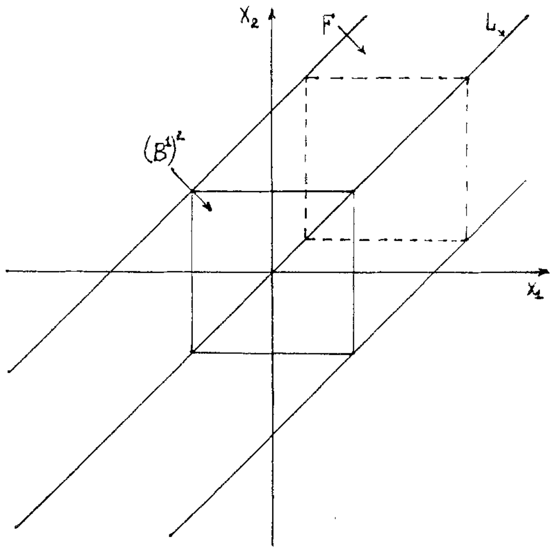

Fig. 1. The strip F is formed by moving the product T of $n+1$ d-dimensional balls B^d along the d-dimensional diagonal L. This figure represents the case when $d=1$, $n=1$, so the balls are just segments and $T=(B^d)^{n+1}$ is a square.

that vertices $x_1^{(d)}, x_2^{(d)},..., x_{n+1}^{(d)}$ form an n-dimensional simplex in $\mathcal{N}$ if and only if the point $(x_1^{(d)}, x_2^{(d)},..., x_{n+1}^{(d)})$ corresponding to them in $\mathbb{R}^{d(n+1)}$ lies in F. Let L_v be the image of V on the diagonal L via the diagonal inclusion $x \mapsto (x,..., x)$. Let us denote by $F_v \subseteq F$ the figure obtained by the moving T along L_v. Then, as $V \to \infty$, we have

$$C_{n,d}(\alpha) = \frac{1}{(n+1)!} \frac{F_v \rho^{n+1}}{V\rho}$$

So the problem is reduced to the calculation of the ratio of the volume of the figure F_v covered by moving T along the diagonal and the magnitude of displacement V. I assert that this ratio is equal to $(n+1)^d [V_d(r)]^n$. I present the proof in the case of $d=1$ below; in the general case the result is obtained by rather cumbersome induction on n and d.

When $d=1$, T is an $(n+1)$-dimensional hypercube and F_v is the union of the sets covered by its forward facets, i.e., the facets of the kind $(x_1,..., r,..., x_{n+1})$. Every facet covers the volume $[V_1(r)]^n V$, and the total number of such facets is $n+1$. That is exactly what we needed.

Now we can return to the proof of the theorem. The equality $\chi_1(\alpha) = e^{-\alpha}$ follows directly from the lemma when $d=1$, and the equality

$$\chi_d(\alpha) = \frac{d}{d\alpha} [\alpha \chi_{d-1}(\alpha)]$$

is just an implication of the analogous equality

$$C_{n,d}(\alpha) = \frac{d}{d\alpha} [\alpha C_{n,d-1}(\alpha)]$$

The theorem is proven.

The proof of the lemma given above can be easily generalized to the case where the initial figure is not a ball but a direct product of several different balls of total dimension d. In this case the sets F_v and V are direct products of the corresponding sets for each ball, and the ratio F_v/V is a product of the corresponding ratios. So the theorem is true in this case, too. Apparently, the theorem remains true for the wide class of initial figures.

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